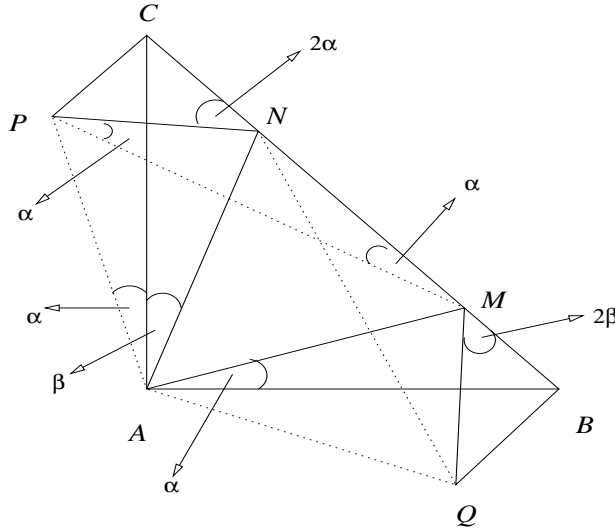


### Solutions to CRMO-2003

1. Let  $ABC$  be a triangle in which  $AB = AC$  and  $\angle CAB = 90^\circ$ . Suppose  $M$  and  $N$  are points on the hypotenuse  $BC$  such that  $BM^2 + CN^2 = MN^2$ . Prove that  $\angle MAN = 45^\circ$ .

**Solution:**

Draw  $CP$  perpendicular to  $CB$  and  $BQ$  perpendicular to  $CB$  such that  $CP = BM$ ,  $BQ = CN$ . Join  $PA, PM, PN, QA, QM, QN$ . (See Fig. 1.)



**Fig. 1.**

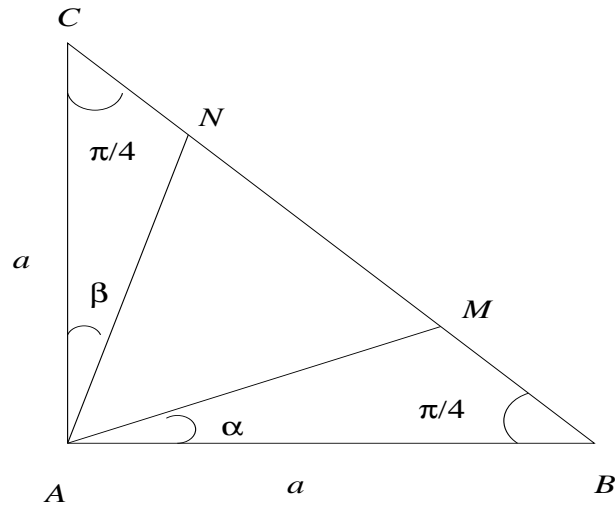
In triangles  $CPA$  and  $BMA$ , we have  $\angle PCA = 45^\circ = \angle MBA$ ;  $PC = MB$ ,  $CA = BA$ . So  $\triangle CPA \cong \triangle BMA$ . Hence  $\angle PAC = \angle BAM = \alpha$ , say. Consequently,  $\angle MAP = \angle BAC = 90^\circ$ , whence  $PAMC$  is a cyclic quadrilateral. Therefore  $\angle PMC = \angle PAC = \alpha$ . Again  $PN^2 = PC^2 + CN^2 = BM^2 + CN^2 = MN^2$ . So  $PN = MN$ , giving  $\angle NPM = \angle NMP = \alpha$ , in  $\triangle PMN$ . Hence  $\angle PNC = 2\alpha$ . Likewise  $\angle QMB = 2\beta$ , where  $\beta = \angle CAN$ . Also  $\triangle NCP \cong \triangle QBM$ , as  $CP = BM$ ,  $NC = BQ$  and  $\angle NCP = 90^\circ = \angle QBM$ . Therefore,  $\angle CPN = \angle BMQ = 2\beta$ , whence  $2\alpha + 2\beta = 90^\circ$ ;  $\alpha + \beta = 45^\circ$ ; finally  $\angle MAN = 90^\circ - (\alpha + \beta) = 45^\circ$ .

**Aliter:** Let  $AB = AC = a$ , so that  $BC = \sqrt{2}a$ ; and  $\angle MAB = \alpha$ ,  $\angle CAN = \beta$ . (See Fig. 2.)

By the Sine Law, we have from  $\triangle ABM$  that

$$\frac{BM}{\sin \alpha} = \frac{AB}{\sin(\alpha + 45^\circ)}.$$

So  $BM = \frac{a\sqrt{2}\sin\alpha}{\cos\alpha + \sin\alpha} = \frac{a\sqrt{2}u}{1+u}$ , where  $u = \tan\alpha$ .



**Fig. 2.**

Similarly  $CN = \frac{a\sqrt{2}v}{1+v}$ , where  $v = \tan\beta$ . But

$$\begin{aligned} BM^2 + CN^2 &= MN^2 = (BC - MB - NC)^2 \\ &= BC^2 + BM^2 + CN^2 \\ &\quad - 2BC \cdot MB - 2BC \cdot NC + MB \cdot NC. \end{aligned}$$

So

$$BC^2 - 2BC \cdot MB - 2BC \cdot NC + 2MB \cdot NC = 0.$$

This reduces to

$$2a^2 - 2\sqrt{2}a \frac{a\sqrt{2}u}{1+u} - 2\sqrt{2}a \frac{a\sqrt{2}v}{1+v} + \frac{4a^2uv}{(1+u)(1+v)} = 0.$$

Multiplying by  $(1+u)(1+v)/2a^2$ , we obtain

$$(1+u)(1+v) - 2u(1+v) - 2v(1+u) + 2uv = 0.$$

Simplification gives  $1 - u - v - uv = 0$ . So

$$\tan(\alpha + \beta) = \frac{u+v}{1-uv} = 1.$$

This gives  $\alpha + \beta = 45^\circ$ , whence  $\angle MAN = 45^\circ$ , as well.

2. If  $n$  is an integer greater than 7, prove that  $\binom{n}{7} - \left[\frac{n}{7}\right]$  is divisible by 7. [Here  $\binom{n}{7}$  denotes the number of ways of choosing 7 objects from among  $n$  objects; also, for any real number  $x$ ,  $[x]$  denotes the greatest integer not exceeding  $x$ .]

**Solution:** We have

$$\binom{n}{7} = \frac{n(n-1)(n-2)\dots(n-6)}{7!}.$$

In the numerator, there is a factor divisible by 7, and the other six factors leave the remainders 1,2,3,4,5,6 in some order when divided by 7.

Hence the numerator may be written as

$$7k \cdot (7k_1 + 1) \cdot (7k_2 + 2) \dots (7k_6 + 6).$$

Also we conclude that  $\left[\frac{n}{p}\right] = k$ , as in the set  $\{n, n-1, \dots, n-6\}$ ,  $7k$  is the only number which is a multiple of 7. If the given number is called  $Q$ , then

$$\begin{aligned} Q &= 7k \cdot \frac{(7k_1 + 1)(7k_2 + 2) \dots (7k_6 + 6)}{7!} - k \\ &= k \left[ \frac{(7k_1 + 1) \dots (7k_6 + 6) - 6!}{6!} \right] \\ &= \frac{k[7t + 6! - 6!]}{6!} \\ &= \frac{7tk}{6!}. \end{aligned}$$

We know that  $Q$  is an integer, and so  $6!$  divides  $7tk$ . Since  $\gcd(7, 6!) = 1$ , even after cancellation there is a factor of 7 still left in the numerator. Hence 7 divides  $Q$ , as desired.

3. Let  $a, b, c$  be three positive real numbers such that  $a + b + c = 1$ . Prove that among the three numbers  $a - ab, b - bc, c - ca$  there is one which is at most  $1/4$  and there is one which is at least  $2/9$ .

**Solution:** By AM-GM inequality, we have

$$a(1-a) \leq \left(\frac{a+1-a}{2}\right)^2 = \frac{1}{4}.$$

Similarly we also have

$$b(1-b) \leq \frac{1}{4} \quad \text{and} \quad c(1-c) \leq \frac{1}{4}.$$

Multiplying these we obtain

$$abc(1-a)(1-b)(1-c) \leq \frac{1}{4^3}.$$

We may rewrite this in the form

$$a(1-b) \cdot b(1-c) \cdot c(1-a) \leq \frac{1}{4^3}.$$

Hence one factor at least (among  $a(1-b), b(1-c), c(1-a)$ ) has to be less than or equal to  $\frac{1}{4}$ ; otherwise **lhs** would exceed  $\frac{1}{4^3}$ .

Again consider the sum  $a(1-b)+b(1-c)+c(1-a)$ . This is equal to  $a+b+c-ab-bc-ca$ . We observe that

$$3(ab+bc+ca) \leq (a+b+c)^2,$$

which, in fact, is equivalent to  $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$ . This leads to the inequality

$$a+b+c-ab-bc-ca \geq (a+b+c) - \frac{1}{3}(a+b+c)^2 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence one summand at least (among  $a(1-b), b(1-c), c(1-a)$ ) has to be greater than or equal to  $\frac{2}{9}$ ; (otherwise **lhs** would be less than  $\frac{2}{3}$ .)

4. Find the number of ordered triples  $(x, y, z)$  of nonnegative integers satisfying the conditions:

- (i)  $x \leq y \leq z$ ;
- (ii)  $x + y + z \leq 100$ .

**Solution:** We count by brute force considering the cases  $x = 0, x = 1, \dots, x = 33$ . Observe that the least value  $x$  can take is zero, and its largest value is 33.

**x = 0** If  $y = 0$ , then  $z \in \{0, 1, 2, \dots, 100\}$ ; if  $y=1$ , then  $z \in \{1, 2, \dots, 99\}$ ; if  $y = 2$ , then  $z \in \{2, 3, \dots, 98\}$ ; and so on. Finally if  $y = 50$ , then  $z \in \{50\}$ . Thus there are altogether  $101 + 99 + 97 + \dots + 1 = 51^2$  possibilities.

**x = 1**. Observe that  $y \geq 1$ . If  $y = 1$ , then  $z \in \{1, 2, \dots, 98\}$ ; if  $y = 2$ , then  $z \in \{2, 3, \dots, 97\}$ ; if  $y = 3$ , then  $z \in \{3, 4, \dots, 96\}$ ; and so on. Finally if  $y = 49$ , then  $z \in \{49, 50\}$ . Thus there are altogether  $98 + 96 + 94 + \dots + 2 = 49 \cdot 50$  possibilities.

**General case.** Let  $x$  be even, say,  $x = 2k$ ,  $0 \leq k \leq 16$ . If  $y = 2k$ , then  $z \in \{2k, 2k+1, \dots, 100-4k\}$ ; if  $y = 2k+1$ , then  $z \in \{2k+1, 2k+2, \dots, 99-4k\}$ ; if  $y = 2k+2$ , then  $z \in \{2k+2, 2k+3, \dots, 99-4k\}$ ; and so on.

Finally, if  $y = 50 - k$ , then  $z \in \{50 - k\}$ . There are altogether

$$(101 - 6k) + (99 - 6k) + (97 - 6k) + \dots + 1 = (51 - 3k)^2$$

possibilities.

Let  $x$  be odd, say,  $x = 2k + 1$ ,  $0 \leq k \leq 16$ . If  $y = 2k + 1$ , then  $z \in \{2k + 1, 2k + 2, \dots, 98 - 4k\}$ ; if  $y = 2k + 2$ , then  $z \in \{2k + 2, 2k + 3, \dots, 97 - 4k\}$ ; if  $y = 2k + 3$ , then  $z \in \{2k + 3, 2k + 4, \dots, 96 - 4k\}$ ; and so on.

Finally, if  $y = 49 - k$ , then  $z \in \{49 - k, 50 - k\}$ . There are altogether

$$(98 - 6k) + (96 - 6k) + (94 - 6k) + \dots + 2 = (49 - 3k)(50 - 3k)$$

possibilities.

The last two cases would be as follows:

$x = 32$ : if  $y = 32$ , then  $z \in \{32, 33, 34, 35, 36\}$ ; if  $y = 33$ , then  $z \in \{33, 34, 35\}$ ; if  $y = 34$ , then  $z \in \{34\}$ ; altogether  $5 + 3 + 1 = 9 = 3^2$  possibilities.

$x = 33$ : if  $y = 33$ , then  $z \in \{33, 34\}$ ; only  $2 = 1 \cdot 2$  possibilities.

Thus the total number of triples, say  $T$ , is given by,

$$T = \sum_{k=0}^{16} (51 - 3k)^2 + \sum_{k=0}^{16} (49 - 3k)(50 - 3k).$$

Writing this in the reverse order, we obtain

$$\begin{aligned} T &= \sum_{k=1}^{17} (3k)^2 + \sum_{k=0}^{17} (3k - 2)(3k - 1) \\ &= 18 \sum_{k=1}^{17} k^2 - 9 \sum_{k=1}^{17} k + 34 \\ &= 18 \left( \frac{17 \cdot 18 \cdot 35}{6} \right) - 9 \left( \frac{17 \cdot 18}{2} \right) + 34 \\ &= 30,787. \end{aligned}$$

Thus the answer is 30787.

### Aliter

It is known that the number of ways in which a given positive integer  $n \geq 3$  can be expressed as a sum of three positive integers  $x, y, z$  (that is,  $x + y + z = n$ ), subject to the condition  $x \leq y \leq z$  is  $\left\{ \frac{n^2}{12} \right\}$ , where  $\{a\}$  represents the integer closest to  $a$ . If

zero values are allowed for  $x, y, z$  then the corresponding count is  $\left\{ \frac{(n+3)^2}{12} \right\}$ , where now  $n \geq 0$ .

Since in our problem  $n = x + y + z \in \{0, 1, 2, \dots, 100\}$ , the desired answer is

$$\sum_{n=0}^{100} \left\{ \frac{(n+3)^2}{12} \right\}.$$

For  $n = 0, 1, 2, 3, \dots, 11$ , the corrections for  $\{ \}$  to get the nearest integers are

$$\frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}, \frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}.$$

So, for 12 consecutive integer values of  $n$ , the sum of the corrections is equal to

$$\left( \frac{3 - 4 - 1 - 0 - 1 - 4 - 3}{12} \right) \times 2 = \frac{-7}{6}.$$

Since  $\frac{101}{12} = 8 + \frac{5}{12}$ , there are 8 sets of 12 consecutive integers in  $\{3, 4, 5, \dots, 103\}$  with 99, 100, 101, 102, 103 still remaining. Hence the total correction is

$$\left( \frac{-7}{6} \right) \times 8 + \frac{3 - 4 - 1 - 0 - 1}{12} = \frac{-28}{3} - \frac{1}{4} = \frac{-115}{12}.$$

So the desired number  $T$  of triples  $(x, y, z)$  is equal to

$$\begin{aligned} T &= \sum_{n=0}^{100} \frac{(n+3)^2}{12} - \frac{115}{12} \\ &= \frac{(1^2 + 2^2 + 3^2 + \dots + 103^2) - (1^2 + 2^2)}{12} - \frac{115}{12} \\ &= \frac{103 \cdot 104 \cdot 207}{6 \cdot 12} - \frac{5}{12} - \frac{115}{12} \\ &= 30787. \end{aligned}$$

5. Suppose  $P$  is an interior point of a triangle  $ABC$  such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. [Here  $d(X, YZ)$  denotes the perpendicular distance from a point  $X$  to the line  $YZ$ .]

**Solution:** Let  $AP, BP, CP$  when extended, meet the sides  $BC, CA, AB$  in  $D, E, F$  respectively. Draw  $AK, PL$  perpendicular to  $BC$  with  $K, L$  on  $BC$ . (See Fig. 3.)

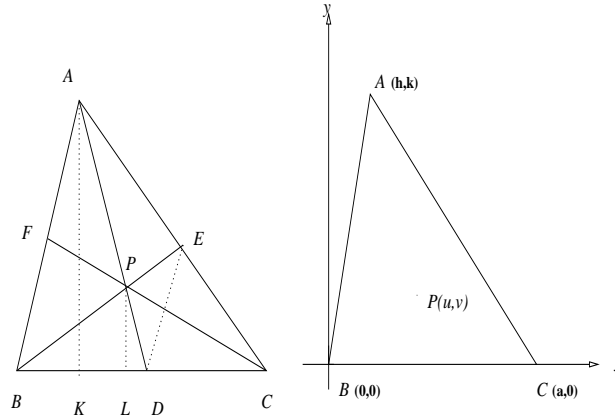


Fig. 3.

Fig. 4.

Now

$$\frac{d(A, BC)}{d(P, BC)} = \frac{AK}{PL} = \frac{AD}{PD}.$$

Similarly,

$$\frac{d(B, CA)}{d(P, CA)} = \frac{BE}{PE} \quad \text{and} \quad \frac{d(C, AB)}{d(P, AB)} = \frac{CF}{PF}.$$

So, we obtain

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF}, \quad \text{and hence} \quad \frac{AP}{PD} = \frac{BP}{PE} = \frac{CP}{PF}.$$

From  $\frac{AP}{PD} = \frac{BP}{PE}$  and  $\angle APB = \angle DPE$ , it follows that triangles  $APB$  and  $DPE$  are similar. So  $\angle ABP = \angle DEP$  and hence  $AB$  is parallel to  $DE$ .

Similarly,  $BC$  is parallel to  $EF$  and  $CA$  is parallel to  $DF$ . Using these we obtain

$$\frac{BD}{DC} = \frac{AE}{EC} = \frac{AF}{FB} = \frac{DC}{BD},$$

whence  $BD^2 = CD^2$  or which is same as  $BD = CD$ . Thus  $D$  is the midpoint of  $BC$ . Similarly  $E, F$  are the midpoints of  $CA$  and  $AB$  respectively.

We infer that  $AD, BE, CF$  are indeed the medians of the triangle  $ABC$  and hence  $P$  is the centroid of the triangle. So

$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} = 3,$$

and consequently each of the given ratios is also equal to 3.

Aliter

Let  $ABC$ , the given triangle be placed in the  $xy$ -plane so that  $B = (0, 0), C = (a, 0)$  (on the  $x$ - axis). (See Fig. 4.)

Let  $A = (h, k)$  and  $P = (u, v)$ . Clearly  $d(A, BC) = k$  and  $d(P, BC) = v$ , so that

$$\frac{d(A, BC)}{d(P, BC)} = \frac{k}{v}.$$

The equation to  $CA$  is  $kx - (h - a)y - ka = 0$ . So

$$\begin{aligned} \frac{d(B, CA)}{d(P, CA)} &= \frac{-ka}{\sqrt{k^2 + (h - a)^2}} \bigg/ \frac{(ku - (h - a)v - ka)}{\sqrt{k^2 + (h - a)^2}} \\ &= \frac{-ka}{ku - (h - a)v - ka}. \end{aligned}$$

Again the equation to  $AB$  is  $kx - hy = 0$ . Therefore

$$\begin{aligned} \frac{d(C, AB)}{d(P, AB)} &= \frac{ka}{\sqrt{h^2 + k^2}} \bigg/ \frac{(ku - hv)}{\sqrt{h^2 + k^2}} \\ &= \frac{ka}{ku - hv}. \end{aligned}$$

From the equality of these ratios, we get

$$\frac{k}{v} = \frac{-ka}{ku - (h - a)v - ka} = \frac{ka}{ku - hv}.$$

The equality of the first and third ratios gives  $ku - (h + a)v = 0$ . Similarly the equality of second and third ratios gives  $2ku - (2h - a)v = ka$ . Solving for  $u$  and  $v$ , we get

$$u = \frac{h + a}{3}, \quad v = \frac{k}{3}.$$

Thus  $P$  is the centroid of the triangle and each of the ratios is equal to  $\frac{k}{v} = 3$ .

6. Find all real numbers  $a$  for which the equation

$$x^2 + (a - 2)x + 1 = 3|x|$$

has exactly three distinct real solutions in  $x$ .

**Solution:** If  $x \geq 0$ , then the given equation assumes the form,

$$x^2 + (a - 5)x + 1 = 0. \quad \dots(1)$$

If  $x < 0$ , then it takes the form

$$x^2 + (a + 1)x + 1 = 0. \quad \dots(2)$$

For these two equations to have exactly three distinct real solutions we should have



- (I) either  $(a - 5)^2 > 4$  and  $(a + 1)^2 = 4$ ;  
 (II) or  $(a - 5)^2 = 4$  and  $(a + 1)^2 > 4$ .

**Case (I)** From  $(a + 1)^2 = 4$ , we have  $a = 1$  or  $-3$ . But only  $a = 1$  satisfies  $(a - 5)^2 > 4$ . Thus  $a = 1$ . Also when  $a = 1$ , equation (1) has solutions  $x = 2 + \sqrt{3}$ ; and (2) has solutions  $x = -1, -1$ . As  $2 + \sqrt{3} > 0$  and  $-1 < 0$ , we see that  $a = 1$  is indeed a solution.

**Case (II)** From  $(a - 5)^2 = 4$ , we have  $a = 3$  or  $7$ . Both these values of  $a$  satisfy the inequality  $(a + 1)^2 > 4$ . When  $a = 3$ , equation (1) has solutions  $x = 1, 1$  and (2) has the solutions  $x = -2 \pm \sqrt{3}$ . As  $1 > 0$  and  $-2 \pm \sqrt{3} < 0$ , we see that  $a = 3$  is in fact a solution.

When  $a = 7$ , equation (1) has solutions  $x = -1, -1$ , which are negative contradicting  $x \geq 0$ .

Thus  $a = 1, a = 3$  are the two desired values.

7. Consider the set  $X = \{1, 2, 3, \dots, 9, 10\}$ . Find two disjoint nonempty subsets  $A$  and  $B$  of  $X$  such that
- $A \cup B = X$ ;
  - $\text{prod}(A)$  is divisible by  $\text{prod}(B)$ , where for any finite set of numbers  $C$ ,  $\text{prod}(C)$  denotes the product of all numbers in  $C$  ;
  - the quotient  $\text{prod}(A)/\text{prod}(B)$  is as small as possible.

**Solution:** The prime factors of the numbers in set  $\{1, 2, 3, \dots, 9, 10\}$  are 2, 3, 5, 7. Also only  $7 \in X$  has the prime factor 7. Hence it cannot appear in  $B$ . For otherwise, 7 in the denominator would not get canceled. Thus  $7 \in A$ .

Hence

$$\text{prod}(A)/\text{prod}(B) \geq 7.$$

The numbers having prime factor 3 are 3, 6, 9. So 3 and 6 should belong to one of  $A$  and  $B$ , and 9 belongs to the other. We may take  $3, 6 \in A, 9 \in B$ .

Also 5 divides 5 and 10. We take  $5 \in A, 10 \in B$ . Finally we take  $1, 2, 4 \in A, 8 \in B$ . Thus

$$A = \{1, 2, 3, 4, 5, 6, 7\}, \quad B = \{8, 9, 10\},$$

so that

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9 \cdot 10} = 7.$$

Thus 7 is the minimum value of  $\frac{\text{prod}(A)}{\text{prod}(B)}$ . There are other possibilities for  $A$  and  $B$ : e.g., 1 may belong to either  $A$  or  $B$ . We may take  $A = \{3, 5, 6, 7, 8\}$ ,  $B = \{1, 2, 4, 9, 10\}$ .