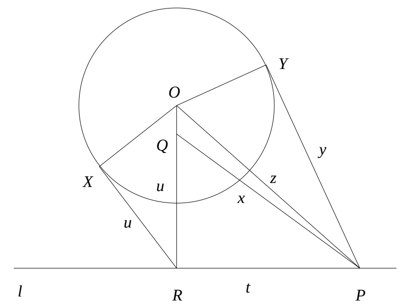
Problems and Solutions

1. Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a unique point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.

   Solution:



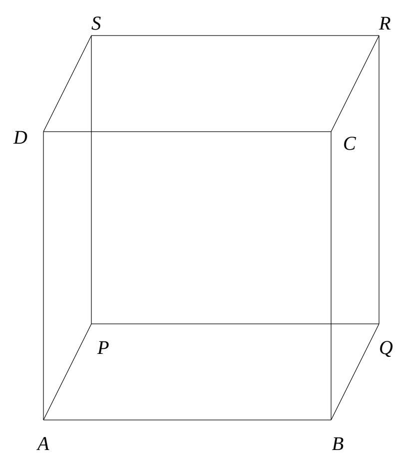
   Let R be the foot of the perpendicular from O to the line l, and u be the length of the tangent RX from R to circle Γ. On OR take a point Q such that $QR = u$. We show that Q is the desired point. To this end, take any point P on line l and let y be the length of the tangent PY from P to Γ.

   Further let r be the radius of the circle Γ and let y be the length of the tangent PY from P to Γ. Join OP, QP. Let $QP = x, OP = z, RP = t$. From right angled triangles POY, OXP, ORP, PQR we have respectively $z^2 = r^2 + y^2, OR^2 = r^2 + u^2, z^2 = OR^2 + t^2 = r^2 + u^2 + t^2, x^2 = u^2 + t^2$. So we obtain $y^2 = z^2 - r^2 = r^2 + u^2 + t^2 - r^2 = u^2 + t^2 = x^2$. Hence $y = x$. This gives $PY = PX$ which is what we needed to show.

2. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

   Solution:

   Let $ABCDPQRS$ be a cube, and the numbers a, b, c, d, e, f be written on the faces $ABCD$, $BQRC$, $PQRS$, $APSD$, $ABQP$, $CRSD$ respectively. Then the products written at the corners A, B, C, D, P, Q, R, S are respectively ade, abe, abf, adf, cde, bce, bcf, cdf. The sum of these 8 numbers is:



$$= \quad (e+f)(ab+bc+cd+ad) = (e+f)(a+c)(b+d).$$

This is given to be equal to $2004 = 2^2 \cdot 3 \cdot 167$. Observe that none of the factors $a+c$, $b+d$, $e+f$ is equal to 1. Thus $(a+c)(b+d)(e+f)$ is equal to $4 \cdot 3 \cdot 167$, $2 \cdot 6 \cdot 167$, $2 \cdot 3 \cdot 334$ or $2 \cdot 2 \cdot 501$. Hence the possible values of $T = a+b+c+d+e+f$ are $4+3+167 = 174$, $2+6+167 = 175$, $2+3+334 = 339$, or $2+2+501 = 505$.

Thus there are 4 possible values of T and they are 174,175,339,505.

3. Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,

   (a) λ_n is an integer; and

   (b) gcd $(\lambda_n, \lambda_{n+1}) = 1$.

   Solution: Since α and β are the roots of the equation $x^2 + mx - 1 = 0$, we have $\alpha^2 + m\alpha - 1 = 0, \beta^2 + m\beta - 1 = 0$. Multiplying by $\alpha^{n-2}, \beta^{n-2}$ respectively we have $\alpha^n + m\alpha^{n-1} - \alpha^{n-2} = 0$ and $\beta^n + m\beta^{n-1} - \beta^{n-2} = 0$.

   Adding we obtain $\alpha^n + \beta^n = -m(\alpha^{n-1} + \beta^{n-1}) + (\alpha^{n-2} + \beta^{n-2})$. This gives a recurrence relation for $n \geq 2$:

   $$\lambda_n = -\lambda_{n-1} + \lambda_{n-2}, n \leq 2 \qquad (\star)$$

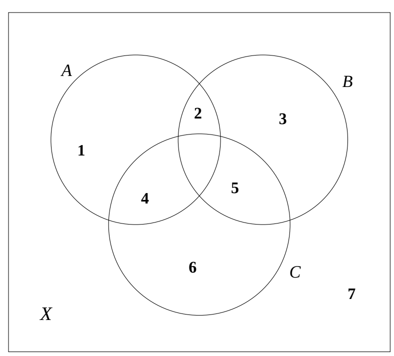
   (a) Now $\lambda_0 = 1 + 1 = 2$ and $\lambda_1 = \alpha + \beta = -m$. Thus λ_0 and λ_1 are integers. By induction, it follows from $(\star)$ that λ_n is an integer for each $n \geq 0$.

   (b) We again use $(\star)$ to prove by induction that gcd $(\lambda_n, \lambda_{n+1}) = 1$. This is clearly true for $n = 0$, as gcd $(2, -m) = 1$, by the given condition that m is odd.

   Let gcd $(\lambda_{n-2}, \lambda_{n-1} = 1, n \geq 2$. If it were to happen that gcd $(\lambda_{n-1}, \lambda_n) > 1$, take a prime p that divides both λ_{n-1} and λ_n. Then from $(\star)$, we get that p divides λ_{n-2} also. Thus p is a factor of gcd $(\lambda_{n-2}, \lambda_{n-1})$, a contradiction. So gcd $(\lambda_{n-1}, \lambda_n) = 1$. Hence we have gcd $(\lambda_n, \lambda_{n+1}) = 1$, for all $n \geq 0$.

4. Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \cdots, n\}$ such that $A \cap B \cap C = \emptyset, A \cap B \neq \emptyset, B \cap C \neq 0$ is $7^n - 2.6^n + 5^n$.

   Solution:



   Let $X = \{1, 2, 3, \cdots, n\}$. We use Venn diagram for sets A, B, C to solve the problem. The regions other than $A \cap B \cap C$ (which is to be empty) are numbered 1,2,3,4,5,6,7 as shown in the figure; e.g., 1 corresponds to $A \backslash (B \cup C) - A \cap B^c \cup C^c$, 2 corresponds to $A \cap B \backslash C = A \cap B \cap C^c$, 7 corresponds to $X \backslash (A \cup B \cup C) = A^c \cap B^c \cap C^c$, since $A \cap B \cap C = \emptyset$.

Firstly the number of ways of assigning elements of X to the numbers regions without any condition is 7^n. Among these there are cases in which 2 or 5 or both are empty. The number of distributions in which 2 is empty is 6^n. Likewise the number of distributions in which 5 is empty is also 6^n. But then we have subtracted twice the number of distributions in which both the regions 2 and 5 are empty. So to compensate we have to add the number of distributions in which both 2 and 5 are empty. This is 5^n. Hence the desired number of triples (A, B, C) in $7^n - 6^n - 6^n + 5^n = 7^n - 2.6^n + 5^n$.

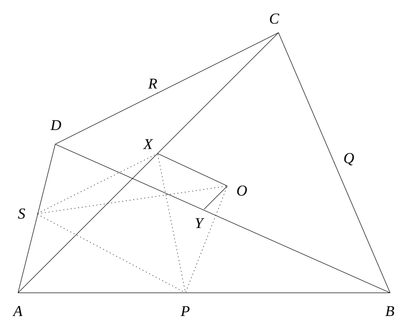
5. Let $ABCD$ be a quadrilateral; x and Y be the midpoints of AC and BD respectively and the lines through X and Y respectively parallel to BD, AC meet in O. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Prove that

   (a) quadrilaterals $APOS$ and $APXS$ have the same area;

   (b) the areas of the quadrilateral $APOS, BQOP, CROQ, DSOR$ are all equal.

Solution:

We use the facts: (i) the line joining the midpoints of the sides of a triangle is parallel to the third side; (ii) any median of a triangle bisects its area; (iii) two triangles having equal bases and bounded by same parallel lines have equal area.



(a) Now BD is parallel to PS as well as OX. So OX is parallel to PS. Hence $[PXS] = [POS]$. Adding $[PAS]$ to both sides we get $[APXS] = [APOS]$. This proves part (a).

(b) Now

$$\begin{aligned}
[APXS] &= [APX] + [ASX] \\
&= \frac{1}{2}[ABX] + \frac{1}{2}[ADX] = \frac{1}{4}[ABC] + \frac{1}{4}[ADC] \\
&= \frac{1}{4}[ABCD].
\end{aligned}$$

Hence by (a), $[APOS] = \frac{1}{4}[ABCD]$. Similarly by symmetry each of the areas $[AQOP], [CROQ]$ and $[DSOR]$ is equal to $\frac{1}{4}[ABCD]$. Thus the four given areas are equal. This proves part (b). [Note: [ ] denotes area].

6. Let $\langle p_1, p_2, p_3, \cdots, p_n, \cdots \rangle$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1, p_{n+1}$ is the largest prime factor of $p_1 p_2 \cdots p_n + 1$. (Thus $p_2 = 3, p_3 = 7$). Prove that $p_n \neq 5$ for any n.

Solution: By data $p_1 = 2, p_2 = 3, p_3 = 7$. It follows by induction that $p_n, n \geq 2$ is odd. [For if $p_2, p_3, \cdots, p_{n-1}$ are odd, then $p_1 p_2 \cdots p_{n-1} + 1$ is also odd and nor 3. This also follows by induction. For if $p_3 = 7$ and if $p_3, p_4, \cdots p_{n-1}$ are neither 2 nor 3, then $p_1 p_2 p_3 \cdots p_{n-1} + 1$ are neither by 2 nor by 3. So p_n is neither 2 nor 3.

7. Let x and y be positive real numbers such that $y^3 + y \le x - x^3$. Prove that

   (a) $y < x < 1$; and

   (b) $x^2 + y^2 < 1$.

Solution:

(a) Since x and y are positive, we have $y \le x - x^3 - y^3 < x$. Also $x - x^3 \ge y + y^3 > 0$. So $x(1-x^2) > 0$. Hence $x < 1$. Thus $y < x < 1$, proving part (a).

(b) Again $x^3 + y^3 \le x - y$. So

$$x^2 - xy + y^2 \le \frac{x-y}{x+y}.$$

That is

$$x^2 + y^2 \le \frac{x-y}{x+y} + xy = \frac{x - y + xy(x+y)}{x+y}.$$

Here $xy(x+y) < 1 \cdot y \cdot (1+1) = 2y$. So $x^2 + y^2 < \dfrac{x-y+2y}{x+y} = \dfrac{x+y}{x+y} = 1$. This proves (b).