1. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are *collinear*.

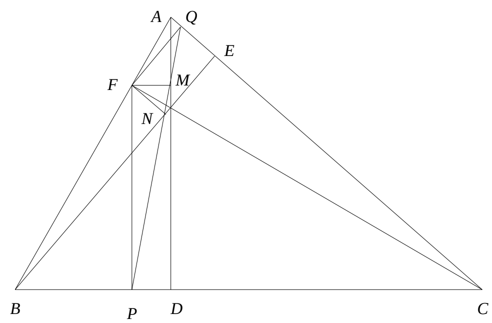
   Solution: Observe that C, Q, F, P are concyclic. Hence

$$\angle CQP = \angle CFP = 90^\circ - \angle FCP = \angle B.$$

   Similarly the concyclicity of F, M, Q, A gives

$$\angle AQN = 90^\circ + \angle FQM = 90^\circ + \angle FAM = 90^\circ + 90^\circ - \angle B = 180^\circ - \angle B.$$

   Thus we obtain $\angle CQP + \angle AQN = 180^\circ$. It follows that Q, N, P lie on the same line.



   We can similarly prove that $\angle CPQ + \angle BPM = 180^\circ$. This implies that P, M, Q are collinear. Thus M, N both lie on the line joining P and Q.

2. Find the *least* possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$.

   Solution: Since 13 divides $a + 11b$, we see that 13 divides $a - 2b$ and hence it also divides $6a - 12b$. This in turn implies that $13|(6a + b)$. Similarly $11|(a + 13b) \Longrightarrow 11|(a + 2b) \Longrightarrow 11|(6a + 12b) \Longrightarrow 11|(6a + b)$. Since $\gcd(11, 13) = 1$, we conclude that $143|(6a + b)$. Thus we may write $6a + b = 143k$ for some natural number k. Hence

$$6a + 6b = 143k + 5b = 144k + 6b - (k + b).$$

   This shows that 6 divides $k + b$ and hence $k + b \geq 6$. We therefore obtain

$$6(a + b) = 143k + 5b = 138k + 5(k + b) \geq 138 + 5 \times 6 = 168.$$

   It follows that $a + b \geq 28$. Taking $a = 23$ and $b = 5$, we see that the conditions of the problem are satisfied. Thus the minimum value of $a + b$ is 28.

3. If a, b, c are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

   Solution: We use the trivial inequalities $a^2 + 1 \geq 2a$, $b^2 + 1 \geq 2b$ and $c^2 + 1 \geq 2c$. Hence we obtain

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b}.$$

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}\geq 3.$$

Adding 6 both sides, this is equivalent to

$$(2a+2b+2c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 9.$$

Taking $x=b+c$, $y=c+a$, $z=a+b$, this is equivalent to

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9.$$

This is a consequence of AM-GM inequality.

Alternately: The substitutions $b+c=x$, $c+a=y$, $a+b=z$ leads to

$$\sum\frac{2a}{b+c}=\sum\frac{y+z-x}{x}=\sum\left(\frac{x}{y}+\frac{y}{x}\right)-3\geq 6-3=3.$$

4. A 6×6 square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always **two** congruent rectangles.

Solution: Consider the dissection of the given 6×6 square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an 1×1 rectangle. Similarly, we get 1×2, 1×3 rectangles for areas 2 ,3 units. In the case of 4 units we may have either a 1×4 rectangle or a 2×2 square. Similarly, there can be a 1×5 rectangle for area 5 units and 1×6 or 2×3 rectangle for 6 units. Any rectangle with area 7 units must be 1×7 rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a 2×4 rectangle If there is any dissection of the given 6×6 square in to 9 non-congruent rectangles with areas $a_1\leq a_2\leq a_3\leq a_4\leq a_5\leq a_6\leq a_7\leq a_8\leq a_9$, then we observe that

$$a_1\geq 1,\ a_2\geq 2,\ a_3\geq 3,\ a_4\geq 4,\ a_5\geq 4,\ a_6\geq 5,\ a_7\geq 6,\ a_8\geq 6,\ a_9\geq 8,$$

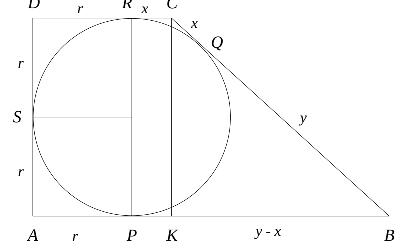
and hence the total area of all the rectangles is

$$a_1+a_2+\cdots+a_9\geq 1+2+3+4+4+5+6+6+8=39>36,$$

which is the area of the given square. Hence if a 6×6 square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

5. Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD; $AB=3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

Solution: Let P, Q, R, S be the points of contact of in-circle with the sides AB, BC, CD, DA respectively. Since AD is perpendicular to AB and AB is parallel to DC, we see that $AP=AS=SD=DR=r$, the radius of the inscribed circle. Let $BP=BQ=y$ and $CQ=CR=x$. Using $AB=3CD$, we get $r+y=3(r+x)$.



Since the area of $ABCD$ is 4, we also get

$$4 = \frac{1}{2}AD(AB + CD) = \frac{1}{2}(2r)\big(4(r + x)\big).$$

Thus we obtain $r(r + x) = 1$. Using Pythagoras theorem, we obtain $BC^2 = BK^2 + CK^2$. However $BC = y + x$, $BK = y - x$ and $CK = 2r$. Substituting these and simplifying, we get $xy = r^2$. But $r+y = 3(r+x)$ gives $y = 2r+3x$. Thus $r^2 = x(2r+3x)$ and this simplifies to $(r - 3x)(r + x) = 0$. We conclude that $r = 3x$. Now the relation $r(r + x) = 1$ implies that $4r^2 = 3$, giving $r = \sqrt{3}/2$.

6. Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in *at least* two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

Solution: Let $Q = n(n + 1)$. It is convenient to choose $n = m^2$, for then Q is already a sum of two squares: $Q = m^2(m^2 + 1) = \big(m^2\big)^2 + m^2$. If further m^2 itself is a sum of two squares, say $m^2 = p^2 + q^2$, then

$$Q = \big(p^2 + q^2\big)\big(m^2 + 1\big) = \big(pm + q\big)^2 + \big(p - qm\big)^2.$$

Note that the two representations for Q are distinct. Thus, for example, we may take $m = 5k$, $p = 3k$, $q = 4k$, where k varies over natural numbers. In this case $n = m^2 = 25k^2$, and

$$Q = \big(25k^2\big)^2 + \big(5k\big)^2 = \big(15k^2 + 4k\big)^2 + \big(20k^2 - 3k\big)^2.$$

As we vary k over natural numbers, we get infinitely many numbers of the from $n(n + 1)$ each of which can be expressed as a sum of two squares in two distinct ways.

7. Let X be the set of all positive integers greater than or equal to 8 and let $f : X \to X$ be a function such that $f(x + y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

Solution: We observe that

$$\begin{aligned} f(9) = f(4 + 5) = f(4 \cdot 5) = f(20) = f(16 + 4) = f(16 \cdot 4) = f(64) \\ = f(8 \cdot 8) = f(8 + 8) = f(16) = f(4 \cdot 4) = f(4 + 4) = f(8). \end{aligned}$$

Hence if $f(8) = 9$, then $f(9) = 9$. (This is one string. There may be other different ways of approaching $f(8)$ from $f(9)$. The important thing to be observed is the fact that the rule $f(x + y) = f(xy)$ applies only when x and y are at least 4. One may get strings using numbers x and y which are smaller than 4, but that is not valid. For example

$$f(9) = f(3 \cdot 3) = f(3 + 3) = f(6) = f(4 + 2) = f(4 \cdot 2) = f(8),$$

is not a valid string.)