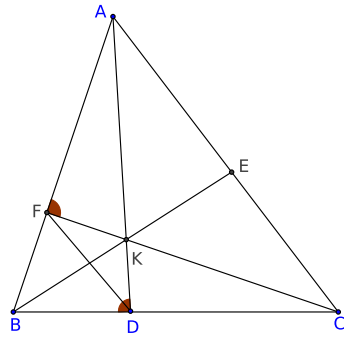


## Problems and Solutions: CRMO-2011

1. Let  $ABC$  be a triangle. Let  $D, E, F$  be points respectively on the segments  $BC, CA, AB$  such that  $AD, BE, CF$  concur at the point  $K$ . Suppose  $BD/DC = BF/FA$  and  $\angle ADB = \angle AFC$ . Prove that  $\angle ABE = \angle CAD$ .



**Solution:** Since  $BD/DC = BF/FA$ , the lines  $DF$  and  $CA$  are parallel. We also have  $\angle BDK = \angle ADB = \angle AFC = 180^\circ - \angle BFK$ , so that  $BDKF$  is a cyclic quadrilateral. Hence  $\angle FBK = \angle FDK$ . Finally, we get

$$\begin{aligned}\angle ABE &= \angle FBK = \angle FDK \\ &= \angle FDA = \angle DAC,\end{aligned}$$

since  $FD \parallel AC$ .

2. Let  $(a_1, a_2, a_3, \dots, a_{2011})$  be a permutation (that is a rearrangement) of the numbers  $1, 2, 3, \dots, 2011$ . Show that there exist two numbers  $j, k$  such that  $1 \leq j < k \leq 2011$  and  $|a_j - j| = |a_k - k|$ .

**Solution:** Observe that  $\sum_{j=1}^{2011} (a_j - j) = 0$ , since  $(a_1, a_2, a_3, \dots, a_{2011})$  is a permutation of  $1, 2, 3, \dots, 2011$ . Hence  $\sum_{j=1}^{2011} |a_j - j|$  is even. Suppose  $|a_j - j| \neq |a_k - k|$  for all  $j \neq k$ . This means the collection  $\{|a_j - j| : 1 \leq j \leq 2011\}$  is the same as the collection  $\{0, 1, 2, \dots, 2010\}$  as the maximum difference is  $2011-1=2010$ . Hence

$$\sum_{j=1}^{2011} |a_j - j| = 1 + 2 + 3 + \dots + 2010 = \frac{2010 \times 2011}{2} = 2011 \times 1005,$$

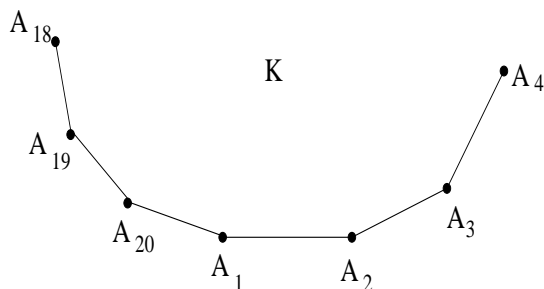
which is odd. This shows that  $|a_j - j| = |a_k - k|$  for some  $j \neq k$ .

3. A natural number  $n$  is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting  $k$  from  $n$  and the larger one is obtained by adding  $l$  to  $n$ . Prove that  $n - kl$  is a perfect square.

**Solution:** Let  $u$  be a natural number such that  $u^2 < n < (u+1)^2$ . Then  $n - k = u^2$  and  $n + l = (u+1)^2$ . Thus

$$\begin{aligned}n - kl &= n - (n - u^2)((u+1)^2 - n) \\ &= n - n(u+1)^2 + n^2 + u^2(u+1)^2 - nu^2 \\ &= n^2 + n(1 - (u+1)^2 - u^2) + u^2(u+1)^2 \\ &= n^2 + n(1 - 2u^2 - 2u - 1) + u^2(u+1)^2 \\ &= n^2 - 2nu(u+1) + (u(u+1))^2 \\ &= (n - u(u+1))^2.\end{aligned}$$

4. Consider a 20-sided convex polygon  $K$ , with vertices  $A_1, A_2, \dots, A_{20}$  in that order. Find the number of ways in which three sides of  $K$  can be chosen so that every pair among them has at least two sides of  $K$  between them. (For example  $(A_1A_2, A_4A_5, A_{11}A_{12})$  is an admissible triple while  $(A_1A_2, A_4A_5, A_{19}A_{20})$  is not.)



**Solution:** First let us count all the admissible triples having  $A_1A_2$  as one of the sides. Having chosen  $A_1A_2$ , we cannot choose  $A_2A_3$ ,  $A_3A_4$ ,  $A_{20}A_1$  nor  $A_{19}A_{20}$ . Thus we have to choose two sides separated by 2 sides among 15 sides  $A_4A_5, A_5A_6, \dots, A_{18}A_{19}$ . If  $A_4A_5$  is one of them, the choice for the remaining side is only from 12 sides

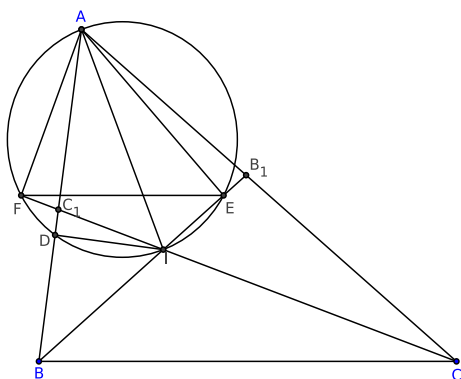
$A_7A_8, A_8A_9, \dots, A_{18}A_{19}$ . If we choose  $A_5A_6$  after  $A_1A_2$ , the choice for the third side is now only from  $A_8A_9, A_9A_{10}, \dots, A_{18}A_{19}$  (11 sides). Thus the number of choices progressively decreases and finally for the side  $A_{15}A_{16}$  there is only one choice, namely,  $A_{18}A_{19}$ . Hence the number of triples with  $A_1A_2$  as one of the sides is

$$12 + 11 + 10 + \dots + 1 = \frac{12 \times 13}{2} = 78.$$

Hence the number of triples then would be  $(78 \times 20)/3 = 520$ .

**Remark:** For an  $n$ -sided polygon, the number of such triples is  $\frac{n(n-7)(n-8)}{6}$ , for  $n \geq 9$ . We may check that for  $n = 20$ , this gives  $(20 \times 13 \times 12)/6 = 520$ .

5. Let  $ABC$  be a triangle and let  $BB_1, CC_1$  be respectively the bisectors of  $\angle B, \angle C$  with  $B_1$  on  $AC$  and  $C_1$  on  $AB$ . Let  $E, F$  be the feet of perpendiculars drawn from  $A$  onto  $BB_1, CC_1$  respectively. Suppose  $D$  is the point at which the incircle of  $ABC$  touches  $AB$ . Prove that  $AD = EF$ .



**Solution:** Observe that  $\angle ADI = \angle AFI = \angle AEI = 90^\circ$ . Hence  $A, F, D, I, E$  all lie on the circle with  $AI$  as diameter. We also know

$$\angle BIC = 90^\circ + \frac{\angle A}{2} = \angle FIE.$$

This gives

$$\begin{aligned} \angle FAE &= 180^\circ - \left(90^\circ + \frac{\angle A}{2}\right) \\ &= 90^\circ - \frac{\angle A}{2}. \end{aligned}$$

We also have  $\angle AID = 90^\circ - \frac{\angle A}{2}$ . Thus  $\angle FAE = \angle AID$ . This shows the chords  $FE$  and  $AD$  subtend equal angles at the circumference of the same circle. Hence they have equal lengths, i.e.,  $FE = AD$ .

6. Find all pairs  $(x, y)$  of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

**Solution:** Observe that

$$x^2 + y + x + y^2 + \frac{1}{2} = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \geq 0.$$

This shows that  $x^2 + y + x + y^2 \geq (-1/2)$ . Hence we have

$$\begin{aligned} 1 = 16^{x^2+y} + 16^{x+y^2} &\geq 2 \left(16^{x^2+y} \cdot 16^{x+y^2}\right)^{1/2}, \quad (\text{by AM-GM inequality}) \\ &= 2 \left(16^{x^2+y+x+y^2}\right)^{1/2} \\ &\geq 2(16)^{-1/4} = 1. \end{aligned}$$

Thus equality holds every where. We conclude that

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 0.$$

This shows that  $(x, y) = (-1/2, -1/2)$  is the only solution, as can easily be verified.

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