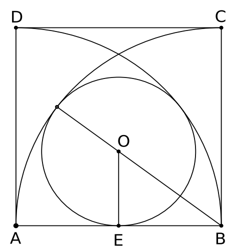
Problems and Solutions: CRMO-2012, Paper 2

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arc AC internally, the arc BD internally and also touching the side AB. Find the radius of the circle Γ.



Solution: Let O be the centre of Γ. By symmetry O is on the perpendicular bisector of AB. Draw $OE \perp AB$. Then $BE = AB/2 = 1/2$. If r is the radius of Γ, we see that $OB = 1 - r$, and $OE = r$. Using Pythagoras' theorem

$$(1-r)^2 = r^2 + \left(\tfrac{1}{2}\right)^2.$$

Simplification gives $r = 3/8$.

2. Let a, b, c be positive integers such that a divides b^4, b divides c^4 and c divides a^4. Prove that abc divides $(a+b+c)^{21}$.

Solution: If a prime p divides a, then $p \mid b^4$ and hence $p \mid b$. This implies that $p \mid c^4$ and hence $p \mid c$. Thus every prime dividing a also divides b and c. By symmetry, this is true for b and c as well. We conclude that a, b, c have the same set of prime divisors.

Let $p^x \,||\, a$, $p^y \,||\, b$ and $p^z \,||\, c$. (Here we write $p^x \,||\, a$ to mean $p^x \mid a$ and $p^{x+1} \not| a$.) We may assume $\min\{x, y, z\} = x$. Now $b \mid c^4$ implies that $y \le 4z$; $c \mid a^4$ implies that $z \le 4x$. We obtain

$$y \le 4z \le 16x.$$

Thus $x + y + z \le x + 4x + 16x = 21x$. Hence the maximum power of p that divides abc is $x + y + z \le 21x$. Since x is the minimum among x, y, z, p^x divides a, b, c. Hence p^x divides $a + b + c$. This implies that p^{21x} divides $(a+b+c)^{21}$. Since $x + y + z \le 21x$, it follows that p^{x+y+z} divides $(a+b+c)^{21}$. This is true of any prime p dividing a, b, c. Hence abc divides $(a+b+c)^{21}$.

3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \le 1.$$

Solution: Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$

Now if $a \le b$, then $a^a \le b^a$ and $a^b \le b^b$. If $a \ge b$, then $a^a \ge b^a$ and $a^b \ge b^b$. Hence the product is nonnegative for all positive a and b. It follows that

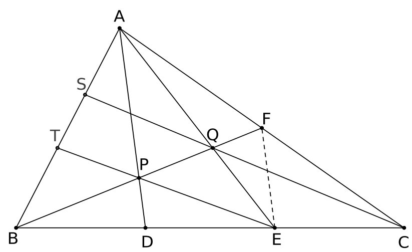
$$a^a b^b + a^b b^a \le 1.$$

4. Let $X = \{1, 2, 3, \ldots, 12\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \ne B$ and $A \cap B = \{2, 3, 5, 7, 8\}$.

Solution: Let $A \cup B = Y$, $B \setminus A = M$, $A \setminus B = N$ and $X \setminus Y = L$. Then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{2, 3, 5, 7, 8\}$ is fixed. The remaining seven elements $1, 4, 6, 9, 10, 11, 12$ can be distributed in any of the remaining sets M, N, L.

This can be done in 3^7 ways. Of these if all the elements are in the set L, then $A = B = \{2, 3, 5, 7, 8\}$ and this case has to be omitted. Hence the total number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7, 8\}$ is $3^7 - 1$.

5. Let ABC be a triangle. Let D, E be a points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC. Let BF intersect AD in P and AE in Q respectively. Determine BP/PQ.



Solution: Let D be the mid-point of BE. Join AD and let it intersect BF in P. Extend CQ and EP to meet AB in S and T respectively. Now

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Similarly,

$$\frac{AQ}{QE} = \frac{[ABQ]}{[EBQ]} = \frac{[ACQ]}{[ECQ]} = \frac{[ABQ]+[ACQ]}{[BCQ]} = \frac{[ABQ]}{[BCQ]} + \frac{[ACQ]}{[BCQ]} = \frac{AF}{FC} + \frac{AS}{SB} = 1 + \frac{1}{2} = \frac{3}{2}.$$

And

$$\frac{AT}{TB} = \frac{[APE]}{[BPE]} = \frac{[APE]}{[APB]} \cdot \frac{[APB]}{[BPE]} = \frac{DE}{DB} \cdot \frac{AQ}{QE} = 1 \cdot \frac{3}{2} = \frac{3}{2}.$$

Finally,

$$\frac{BP}{PQ} = \frac{[BPE]}{[QPE]} = \frac{[BPA]}{[APE]} = \frac{[BPE]+[BPA]}{[APE]} = \frac{[BPE]}{[APE]} + \frac{[BPA]}{[APE]} = \frac{BT}{TA} + \frac{BD}{DE} = \frac{2}{3} + 1 = \frac{5}{3}.$$

(Note: BS/SA, AT/TB can also be obtained using Ceva's theorem. A solution can also be obtained using coordinate geometry.)

6. Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution: Consider the equation whose roots are x, y, z:

$$(t - x)(t - y)(t - z) = 0.$$

This gives $t^3 - 3t - \lambda = 0$, where $\lambda = xyz$. Since x, y, z are roots of this equation, we have

$$x^3 - 3x - \lambda = 0,\ y^3 - 3y - \lambda = 0,\ z^3 - 3z - \lambda = 0.$$

Multiplying the first by y, the second by z and the third by x, we obtain

$$x^3y - 3xy - \lambda y = 0,$$
$$y^3z - 3yz - \lambda z = 0,$$
$$z^3x - 3zx - \lambda x = 0.$$

Adding we obtain

$$x^3y + y^3z + z^3x - 3(xy + yz + zx) - \lambda(x + y + z) = 0.$$

This simplifies to

$$x^3y + y^3z + z^3x = -9.$$

(Here one may also solve for y and z in terms of x and substitute these values in $x^3y + y^3z + z^3x$ to get -9.)

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