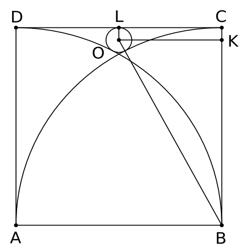
Problems and Solutions: CRMO-2012, Paper 3

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arcs AC and BD both externally and also touching the side CD. Find the radius of the circle Γ.



Solution: Let O be the centre of Γ. By symmetry O is on the perpendicular bisector of CD. Draw $OL \perp CD$ and $OK \perp BC$. Then $OK = CL = CD/2 = 1/2$. If r is the radius of Γ, we see that $BK = 1 - r$, and $OE = r$. Using Pythagoras' theorem

$$(1+r)^2 = (1-r)^2 + \left(\tfrac{1}{2}\right)^2.$$

Simplification gives $r = 1/16$.

2. Let a, b, c be positive integers such that a divides b^5, b divides c^5 and c divides a^5. Prove that abc divides $(a+b+c)^{31}$.

Solution: If a prime p divides a, then $p \mid b^5$ and hence $p \mid b$. This implies that $p \mid c^4$ and hence $p \mid c$. Thus every prime dividing a also divides b and c. By symmetry, this is true for b and c as well. We conclude that a, b, c have the same set of prime divisors.

Let $p^x \mid\mid a$, $p^y \mid\mid b$ and $p^z \mid\mid c$. (Here we write $p^x \mid\mid a$ to mean $p^x \mid a$ and $p^{x+1} \not| a$.) We may assume $\min\{x, y, z\} = x$. Now $b \mid c^5$ implies that $y \leq 5z$; $c \mid a^5$ implies that $z \leq 5x$. We obtain

$$y \leq 5z \leq 25x.$$

Thus $x + y + z \leq x + 5x + 25x = 31x$. Hence the maximum power of p that divides abc is $x + y + z \leq 31x$. Since x is the minimum among x, y, z, p^x divides a, b, c. Hence p^x divides $a + b + c$. This implies that p^{31x} divides $(a+b+c)^{21}$. Since $x + y + z \leq 31x$, it follows that p^{x+y+z} divides $(a+b+c)^{31}$. This is true of any prime p dividing a, b, c. Hence abc divides $(a+b+c)^{31}$.

3. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

Solution: Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$

Now if $a \leq b$, then $a^a \leq b^a$ and $a^b \leq b^b$. If $a \geq b$, then $a^a \geq b^a$ and $a^b \geq b^b$. Hence the product is nonnegative for all poitive a and b. It follows that

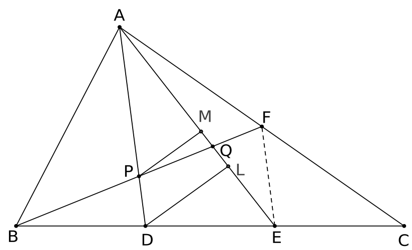
$$a^a b^b + a^b b^a \leq 1.$$

4. Let $X = \{1, 2, 3, \ldots, 10\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{5, 7, 8\}$.

Solution: Let $A \cup B = Y$, $B \setminus A = M$, $A \setminus B = N$ and $X \setminus Y = L$. Then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{5, 7, 8\}$ is fixed. The remaining seven elements $1, 2, 3, 4, 6, 9, 10$ can be distributed in any of the remaining sets M,

N, L. This can be done in 3^7 ways. Of these if all the elements are in the set L, then $A = B = \{5, 7, 8\}$ and this case has to be omitted. Hence the total number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{5, 7, 8\}$ is $3^7 - 1$.

5. Let ABC be a triangle. Let D, E be a points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC. Let BF intersect AD in P and AE in Q respectively. Determine the ratio of the area of the triangle APQ to that of the quadrilateral $PDEQ$.



Solution: If we can find $[APQ]/[ADE]$, then we can get the required ratio as

$$\frac{[APQ]}{[PDEQ]} = \frac{[APQ]}{[ADE] - [APQ]} = \frac{1}{([ADE]/[APQ]) - 1}.$$

Now draw $PM \perp AE$ and $DL \perp AE$. Observe
$[APQ] = \frac{1}{2} AQ \cdot PM$, $[ADE] = \frac{1}{2} AE \cdot DL$.

Further, since $PM \parallel DL$, we also get $PM/DL = AP/AD$. Using these we obtain

$$\frac{[APQ]}{[ADE]} = \frac{AP}{AD} \cdot \frac{AQ}{AE}.$$

We have

$$\frac{AQ}{QE} = \frac{[ABQ]}{[EBQ]} = \frac{[ACQ]}{[ECQ]} = \frac{[ABQ] + [ACQ]}{[BCQ]} = \frac{[ABQ]}{[BCQ]} + \frac{[ACQ]}{[BCQ]} = \frac{AF}{FC} + \frac{AS}{SB}.$$

However

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Besides $AF/FC = 1$. We obtain

$$\frac{AQ}{QE} = \frac{AF}{FC} + \frac{AS}{SB} = 1 + \frac{1}{2} = \frac{3}{2}, \quad \frac{AE}{QE} = 1 + \frac{3}{2} = \frac{5}{2}, \quad \frac{AQ}{AE} = \frac{3}{5}.$$

Since $EF \parallel AD$ (since $DE/EC = AF/FC = 1$), we get $AD = 2EF$. Since $EF \parallel PD$, we also have $PD/EF = BD/DE = 1/2$. Hence $EF = 2PD$. Thus $AD = 4PD$. This gives and $AP/PD = 3$ and $AP/AD = 3/4$. Thus

$$\frac{[APQ]}{[ADE]} = \frac{AP}{AD} \cdot \frac{AQ}{AE} = \frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}.$$

Finally,

$$\frac{[APQ]}{[PDEQ]} = \frac{1}{([ADE]/[APQ]) - 1} = \frac{1}{(20/9) - 1} = \frac{9}{11}.$$

(Note: BS/SA can also be obtained using Ceva's theorem. Coordinate geometry solution can also be obtained.)

6. Find all positive integers n such that $3^{2n} + 3n^2 + 7$ is a perfect square.

Solution: If $3^{2n} + 3n^2 + 7 = b^2$ for some natural number b, then $b^2 > 3^{2n}$ so that $b > 3^n$. This implies that $b \geq 3^n + 1$. Thus

$$3^{2n} + 3n^2 + 7 = b^2 \geq (3^n + 1)^2 = 3^{2n} + 2 \cdot 3^n + 1.$$

This shows that $2 \cdot 3^n \leq 3n^2 + 6$. If $n \geq 3$, this cannot hold. One can prove this eithe by induction or by direct argument:
If $n \geq 3$, then

$$\begin{aligned} 2 \cdot 3^n = 2(1+2)^n = 2\Big(1 + 2n + \big(n(n-1)/2\big) \cdot 2^2 + \cdots\Big) &> 2 + 4n + 4n^2 - 4n \\ &= 3n^2 + (n^2 + 2) \geq 3n^2 + 11 > 3n^2 + 6. \end{aligned}$$

Hence $n = 1$ or 2.
If $n = 1$, then $3^{2n} + 3n^2 + 7 = 19$ and this is not a perfect square. If $n = 2$, we obtain $3^{2n} + 3n^2 + 7 = 81 + 12 + 7 = 100 = 10^2$. Hence $n = 2$ is the only solution.

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