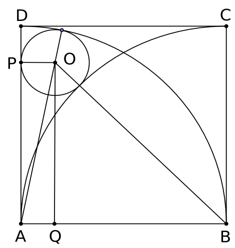
Problems and Solutions: CRMO-2012, Paper 4

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arc AC externally, the arc BD internally and also touching the side AD. Find the radius of the circle Γ.



Solution: Let O be the centre of Γ and r its radius. Draw $OP \perp AD$ and $OQ \perp AB$. Then $OP = r$, $OQ^2 = OA^2 - r^2 = (1-r)^2 - r^2 = 1 - 2r$. We also have $OB = 1 + r$ and $BQ = 1 - r$. Using Pythagoras' theorem we get
$$(1+r)^2 = (1-r)^2 + 1 - 2r.$$
Simplification gives $r = 1/6$.

2. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a+b+c)^7$.

Solution: If a prime p divides a, then $p \mid b^2$ and hence $p \mid b$. This implies that $p \mid c^2$ and hence $p \mid c$. Thus every prime dividing a also divides b and c. By symmetry, this is true for b and c as well. We conclude that a, b, c have the same set of prime divisors.

Let $p^x \,||\, a$, $p^y \,||\, b$ and $p^z \,||\, c$. (Here we write $p^x \,||\, a$ to mean $p^x \mid a$ and $p^{x+1} \not| a$.) We may assume $\min\{x, y, z\} = x$. Now $b \mid c^2$ implies that $y \le 2z$; $c \mid a^2$ implies that $z \le 2x$. We obtain
$$y \le 2z \le 4x.$$
Thus $x + y + z \le x + 2x + 4x = 7x$. Hence the maximum power of p that divides abc is $x + y + z \le 7x$. Since x is the minimum among x, y, z, p^x divides a, b, c. Hence p^x divides $a + b + c$. This implies that p^{7x} divides $(a+b+c)^7$. Since $x + y + z \le 7x$, it follows that p^{x+y+z} divides $(a+b+c)^7$. This is true of any prime p dividing a, b, c. Hence abc divides $(a+b+c)^7$.

3. Let a and b be positive real numbers such that $a + b = 1$. Prove that
$$a^a b^b + a^b b^a \le 1.$$

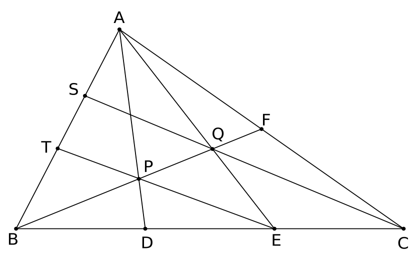
Solution: Observe
$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a b^b.$$
Hence
$$1 - a^a b^b - a^b b^a = a^a b^b + b^a b^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$
Now if $a \le b$, then $a^a \le b^a$ and $a^b \le b^b$. If $a \ge b$, then $a^a \ge b^a$ and $a^b \ge b^b$. Hence the product is nonnegative for all positive a and b. It follows that
$$a^a b^b + a^b b^a \le 1.$$

4. Let $X = \{1, 2, 3, \ldots, 11\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \ne B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

Solution: Let $A \cup B = Y$, $B \setminus A = M$, $A \setminus B = N$ and $X \setminus Y = L$. Then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{4, 5, 7, 8, 9, 10\}$ is fixed. The remaining 5 elements 1, 2, 3, 6, 11 can be distributed in any of the remaining sets M,

N, L. This can be done in 3^5 ways. Of these if all the elements are in the set L, then $A = B = \{4, 5, 7, 8, 9, 10\}$ and this case has to be omitted. Hence the total number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$ is $3^5 - 1$.

5. Let ABC be a triangle. Let E be a point on the segment BC such that $BE = 2EC$. Let F be the mid-point of AC. Let BF intersect AE in Q. Determine BQ/QF.



Solution: Let CQ and ET meet AB in S and T respectively. We have

$$\frac{[SBC]}{[ASC]} = \frac{BS}{SA} = \frac{[SBQ]}{[ASQ]}.$$

Using componendo by dividendo, we obtain

$$\frac{BS}{SA} = \frac{[SBC] - [SBQ]}{[ASC] - [ASQ]} = \frac{[BQC]}{[AQC]}.$$

Similarly, We can prove

$$\frac{BE}{EC} = \frac{[BQA]}{[CQA]}, \quad \frac{CF}{FA} = \frac{[CQB]}{[AQB]}.$$

But $BD = DE = EC$ implies that $BE/EC = 2$; $CF = FA$ gives $CF/FA = 1$. Thus

$$\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2.$$

Now

$$\frac{BQ}{QF} = \frac{[BQC]}{[FQC]} = \frac{[BQA]}{[FQA]} = \frac{[BQC] + [BQA]}{[FQC] + [FQA]} = \frac{[BQC] + [BQA]}{[AQC]}.$$

This gives

$$\frac{BQ}{QF} = \frac{[BQC] + [BQA]}{[AQC]} = \frac{[BQC]}{[AQC]} + \frac{[BQA]}{[AQC]} = \frac{BS}{SA} + \frac{BE}{EC} = 2 + 2 = 4.$$

(Note: BS/SA can also be obtained using Ceva's theorem. One can also obtain the result by coordinate geometry.)

6. Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1.$$

Solution: The given system reduces to

$$z = x^2y + xyz, \; x = y^z + xyz, \; y = z^2x + xyz.$$

Hence

$$z - x^2y = x - y^2z = y - z^2x.$$

If $x = y$, then $y^2z = z^2x$ and hence $x^2z = z^2x$. This implies that $z = x = y$. Similarly, $x = z$ implies that $x = z = y$. Hence if any two of x, y, z are equal, then all are equal. Suppose no two of x, y, z are equal. We may take x is the largest among x, y, z so that $x > y$ and $x > z$. Here we have two possibilities: $y > z$ and $z > y$.

Suppose $x > y > z$. Now $z - x^2y = x - y^2z = y - z^2x$ shows that

$$y^2z > z^2x > x^2y.$$

But $y^2z > z^2x$ and $z^2x > x^2y$ give $y^2 > zx$ and $z^2 > xy$. Hence

$$(y^2)(z^2) > (zx)(xy).$$

This gives $yz > x^2$. Thus $x^3 < xyz = (xz)y < (y^2)y = y^3$. This forces $x < y$ contradicting $x > y$.

Similarly, we arrive at a contradiction if $x > z > y$. The only possibility is $x = y = z$.

For $x = y = z$, we get only one equation $x^2 = 1/2$. Since $x > 0$, $x = 1/\sqrt{2} = y = z$.

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