CHAPTER 1

REAL NUMBERS

(A) Main Concepts and Results

- Euclid's Division Lemma : Given two positive integers a and b, there exist unique integers q and r satisfying $a = bq + r$, $0 \leq r < b$.
- Euclid's Division Algorithm to obtain the HCF of two positive integers, say c and d, $c > d$.

  Step 1 : Apply Euclid's division lemma to c and d, to find whole numbers q and r, such that $c = dq + r$, $0 \leq r < d$.

  Step 2 : If $r = 0$, d is the HCF of c and d. If $r \neq 0$, apply the division lemma to d and r.

  Step 3 : Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.
- Fundamental Theorem of Arithmetic : Every composite number can be expressed as a product of primes, and this expression (factorisation) is unique, apart from the order in which the prime factors occur.
- Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.
- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are irrational numbers.
- The sum or difference of a rational and an irrational number is irrational.
- The product or quotient of a non-zero rational number and an irrational number is irrational.
- For any two positive integers a and b, HCF (a, b) × LCM $(a, b) = a \times b$.

- Let $x = \frac{p}{q}$, p and q are co-prime, be a rational number whose decimal expansion terminates. Then, the prime factorisation of q is of the form $2^m.5^n$; m, n are non-negative integers.
- Let $x = \frac{p}{q}$ be a rational number such that the prime factorisation of q is not of the form $2^m.5^n$; m, n being non-negative integers. Then, x has a non-terminating repeating decimal expansion.

(B) Multiple Choice Questions

Choose the correct answer from the given four options:

Sample Question 1 : The decimal expansion of the rational number $\frac{33}{2^2.5}$ will terminate after

(A) one decimal place (B) two decimal places

(C) three decimal places (D) more than 3 decimal places

Solution : Answer (B)

Sample Question 2 : Euclid's division lemma states that for two positive integers a and b, there exist unique integers q and r such that $a = bq + r$, where r must satisfy

(A) $1 < r < b$ (B) $0 < r \leq b$

(C) $0 \leq r < b$ (D) $0 < r < b$

Solution : Answer (C)

EXERCISE 1.1

Choose the correct answer from the given four options in the following questions:

1. For some integer m, every even integer is of the form

(A) m (B) $m + 1$

(C) $2m$ (D) $2m + 1$

2. For some integer q, every odd integer is of the form

(A) q (B) $q + 1$

(C) $2q$ (D) $2q + 1$

3. $n^2 - 1$ is divisible by 8, if n is

(A) an integer (B) a natural number

(C) an odd integer (D) an even integer

4. If the HCF of 65 and 117 is expressible in the form $65m - 117$, then the value of m is

(A) 4 (B) 2

(C) 1 (D) 3

5. The largest number which divides 70 and 125, leaving remainders 5 and 8, respectively, is

(A) 13 (B) 65

(C) 875 (D) 1750

6. If two positive integers a and b are written as

$a = x^3y^2$ and $b = xy^3$; x, y are prime numbers, then HCF (a, b) is

(A) xy (B) xy^2 (C) x^3y^3 (D) x^2y^2

7. If two positive integers p and q can be expressed as

$p = ab^2$ and $q = a^3b$; a, b being prime numbers, then LCM (p, q) is

(A) ab (B) a^2b^2 (C) a^3b^2 (D) a^3b^3

8. The product of a non-zero rational and an irrational number is

(A) always irrational (B) always rational

(C) rational or irrational (D) one

9. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(A) 10 (B) 100 (C) 504 (D) 2520

10. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after:

(A) one decimal place (B) two decimal places

(C) three decimal places (D) four decimal places

(C) Short Answer Questions with Reasoning

Sample Question 1: The values of the remainder r, when a positive integer a is divided by 3 are 0 and 1 only. Justify your answer.

Solution : No.

According to Euclid's division lemma,

$$a = 3q + r, \text{ where } 0 \leq r < 3$$

and r is an integer. Therefore, the values of r can be 0, 1 or 2.

Sample Question 2: Can the number 6^n, n being a natural number, end with the digit 5? Give reasons.

Solution : No, because $6^n = (2 \times 3)^n = 2^n \times 3^n$, so the only primes in the factorisation of 6^n are 2 and 3, and not 5.

Hence, it cannot end with the digit 5.

EXERCISE 1.2

1. Write whether every positive integer can be of the form $4q + 2$, where q is an integer. Justify your answer.
2. "The product of two consecutive positive integers is divisible by 2". Is this statement true or false? Give reasons.
3. "The product of three consecutive positive integers is divisible by 6". Is this statement true or false"? Justify your answer.
4. Write whether the square of any positive integer can be of the form $3m + 2$, where m is a natural number. Justify your answer.
5. A positive integer is of the form $3q + 1$, q being a natural number. Can you write its square in any form other than $3m + 1$, i.e., $3m$ or $3m + 2$ for some integer m? Justify your answer.
6. The numbers 525 and 3000 are both divisible only by 3, 5, 15, 25 and 75. What is HCF (525, 3000)? Justify your answer.
7. Explain why $3 \times 5 \times 7 + 7$ is a composite number.
8. Can two numbers have 18 as their HCF and 380 as their LCM? Give reasons.
9. Without actually performing the long division, find if $\frac{987}{10500}$ will have terminating or non-terminating (repeating) decimal expansion. Give reasons for your answer.
10. A rational number in its decimal expansion is 327.7081. What can you say about the prime factors of q, when this number is expressed in the form $\frac{p}{q}$? Give reasons.

(D) Short Answer Questions

Sample Question 1: Using Euclid's division algorithm, find which of the following pairs of numbers are co-prime:

(i) 231, 396 (ii) 847, 2160

Solution : Let us find the HCF of each pair of numbers.

(i) $396 = 231 \times 1 + 165$

$231 = 165 \times 1 + 66$

$165 = 66 \times 2 + 33$

$66 = 33 \times 2 + 0$

Therefore, HCF = 33. Hence, numbers are not co-prime.

(ii) $2160 = 847 \times 2 + 466$

$847 = 466 \times 1 + 381$

$466 = 381 \times 1 + 85$

$381 = 85 \times 4 + 41$

$85 = 41 \times 2 + 3$

$41 = 3 \times 13 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

Therefore, the HCF = 1. Hence, the numbers are co-prime.

Sample Question 2: Show that the square of an odd positive integer is of the form $8m + 1$, for some whole number m.

Solution: Any positive odd integer is of the form $2q + 1$, where q is a whole number.

Therefore, $(2q + 1)^2 = 4q^2 + 4q + 1 = 4q\,(q + 1) + 1,$ (1)

$q\,(q + 1)$ is either 0 or even. So, it is $2m$, where m is a whole number.

Therefore, $(2q + 1)^2 = 4.2\,m + 1 = 8\,m + 1.$ [From (1)]

Sample Question 3: Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Solution : Let us suppose that $\sqrt{2}+\sqrt{3}$ is rational. Let $\sqrt{2}+\sqrt{3} = a$, where a is rational.

Therefore, $\sqrt{2} = a - \sqrt{3}$

Squaring on both sides, we get

$$2 = a^2 + 3 - 2a\sqrt{3}$$

Therefore, $\sqrt{3} = \frac{a^2+1}{2a}$, which is a contradiction as the right hand side is a rational number while $\sqrt{3}$ is irrational. Hence, $\sqrt{2}+\sqrt{3}$ is irrational.

EXERCISE 1.3

1. Show that the square of any positive integer is either of the form $4q$ or $4q + 1$ for some integer q.

2. Show that cube of any positive integer is of the form $4m$, $4m + 1$ or $4m + 3$, for some integer m.

3. Show that the square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$ for any integer q.

4. Show that the square of any positive integer cannot be of the form $6m + 2$ or $6m + 5$ for any integer m.

5. Show that the square of any odd integer is of the form $4q + 1$, for some integer q.

6. If n is an odd integer, then show that $n^2 - 1$ is divisible by 8.

7. Prove that if x and y are both odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

8. Use Euclid's division algorithm to find the HCF of 441, 567, 693.

9. Using Euclid's division algorithm, find the largest number that divides 1251, 9377 and 15628 leaving remainders 1, 2 and 3, respectively.

10. Prove that $\sqrt{3}+\sqrt{5}$ is irrational.

11. Show that 12^n cannot end with the digit 0 or 5 for any natural number n.

12. On a morning walk, three persons step off together and their steps measure 40 cm, 42 cm and 45 cm, respectively. What is the minimum distance each should walk so that each can cover the same distance in complete steps?

13. Write the denominator of the rational number $\frac{257}{5000}$ in the form $2^m \times 5^n$, where m, n are non-negative integers. Hence, write its decimal expansion, without actual division.

14. Prove that $\sqrt{p} + \sqrt{q}$ is irrational, where p, q are primes.

(E) Long Answer Questions

Sample Question 1 : Show that the square of an odd positive integer can be of the form $6q + 1$ or $6q + 3$ for some integer q.

Solution : We know that any positive integer can be of the form $6m$, $6m + 1$, $6m + 2$, $6m + 3$, $6m + 4$ or $6m + 5$, for some integer m.

Thus, an odd positive integer can be of the form 6m + 1, 6m + 3, or 6m + 5 Thus we have:

$(6m + 1)^2 = 36m^2 + 12m + 1 = 6(6m^2 + 2m) + 1 = 6q + 1$, q is an integer

$(6m + 3)^2 = 36m^2 + 36m + 9 = 6(6m^2 + 6m + 1) + 3 = 6q + 3$, q is an integer

$(6m + 5)^2 = 36m^2 + 60m + 25 = 6(6m^2 + 10m + 4) + 1 = 6q + 1$, q is an integer.

Thus, the square of an odd positive integer can be of the form $6q + 1$ or $6q + 3$.

EXERCISE 1.4

1. Show that the cube of a positive integer of the form $6q + r$, q is an integer and $r = 0, 1, 2, 3, 4, 5$ is also of the form $6m + r$.
2. Prove that one and only one out of n, $n + 2$ and $n + 4$ is divisible by 3, where n is any positive integer.
3. Prove that one of any three consecutive positive integers must be divisible by 3.
4. For any positive integer n, prove that $n^3 - n$ is divisible by 6.
5. Show that one and only one out of n, $n + 4$, $n + 8$, $n + 12$ and $n + 16$ is divisible by 5, where n is any positive integer.

   [**Hint:** Any positive integer can be written in the form $5q$, $5q+1$, $5q+2$, $5q+3$, $5q+4$].