Problems and Solutions... CRMO-2002

1. In an acute triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ABC.

Solution: Put $CD = x$. Then with usual notations we get

$$CE = \frac{CD \cdot CB}{CA} = \frac{ax}{b}.$$

Since $AE = AC - CE = b - CE$, we obtain

$$AE = \frac{b^2 - ax}{b}, \quad AF = \frac{AE \cdot AC}{AB} = \frac{b^2 - ax}{c}.$$

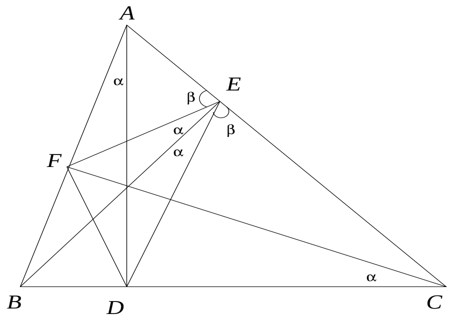

Fig. 1

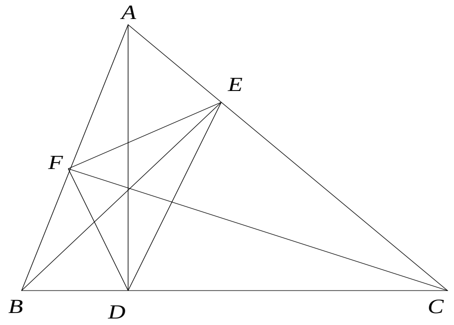

Fig. 2

This in turn gives

$$BF = AB - AF = \frac{c^2 - b^2 + ax}{c}.$$

Finally we obtain

$$BD = \frac{c^2 - b^2 + ax}{a}.$$

Using $BD = a - x$, we get

$$x = \frac{a^2 - c^2 + b^2}{2a}.$$

However, if L is the foot of perpendicular from A on to BC then, using Pythagoras theorem in triangles ALB and ALC we get

$$b^2 - LC^2 = c^2 - (a - LC)^2$$

which reduces to $LC = (a^2 - c^2 + b^2)/2a$. We conclude that $LC = DC$ proving $L = D$. Or, we can also infer that $x = b \cos C$ from cosine rule in triangle ABC. This implies that $CD = CL$, since $CL = b \cos C$ from right triangle ALC. Thus AD is altitude on to BC. Similar proof works for the remaining altitudes.

Alternately, we see that $CD \cdot CB = CE \cdot CA$, so that $ABDE$ is a cyclic quadrilateral. Similarly we infer that $BCEF$ and $CAFD$ are also cyclic quadrilaterals. (See Fig. 2.) Thus $\angle AEF = \angle B = \angle CED$. Moreover $\angle BED = \angle DAF = \angle DCF = \angle BCF = \angle BEF$. It follows that $\angle BEA = \angle BEC$ and hence each is a right angle thus proving that BE is an altitude. Similarly we prove that CF and AD are altitudes. (Note that the concurrence of the lines AD, BE, CF are not required.)

2. Solve the following equation for real x:

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

Solution: By setting $u = x^2 + x - 2$ and $v = 2x^2 - x - 1$, we observe that the equation reduces to $u^3 + v^3 = (u + v)^3$. Since $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, it follows that $uv(u + v) = 0$. Hence $u = 0$ or $v = 0$ or $u + v = 0$. Thus we obtain $x^2 + x - 2 = 0$ or $2x^2 - x - 1 = 0$ or $x^2 - 1 = 0$. Solving each of them we get $x = 1, -2$ or $x = 1, -1/2$ or $x = 1, -1$. Thus $x = 1$ is a root of multiplicity 3 and the other roots are $-1, -2, -1/2$.

(Alternately, it can be seen that $x - 1$ is a factor of $x^2 + x - 2$, $2x^2 - x - 1$ and $x^2 - 1$. Thus we can write the equation in the form

$$(x - 1)^3(x + 2)^3 + (x - 1)^3(2x + 1)^3 = 27(x - 1)^3(x + 1)^3.$$

Thus it is sufficient to solve the cubic equation

$$(x + 2)^3 + (2x + 1)^3 = 27(x + 1)^3.$$

This can be solved as earlier or expanding every thing and simplifying the relation.)

3. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$.

Solution: Consider the expansion of $(a + b + c)^7$. We show that each term here is divisible by abc. It contains terms of the form $r_{klm} a^k b^l c^m$, where r_{klm} is a constant(some binomial coefficient) and k, l, m are nonnegative integers such that $k + l + m = 7$. If $k \geq 1$, $l \geq 1$, $m \geq 1$, then abc divides $a^k b^l c^m$. Hence we have to consider terms in which one or two of k, l, m are zero. Suppose for example $k = l = 0$ and consider c^7. Since b divides c^2 and a divides c^4, it follows that abc divides c^7. A similar argument gives the result for a^7 or b^7. Consider the case in which two indices are nonzero, say for example, bc^6. Since a divides c^4, here again abc divides bc^6. If we take b^2c^5, then also using a divides c^4 we obtain the result. For b^3c^4, we use the fact that a divides b^2. Similar argument works for b^4c^3, b^5c^2 and b^6c. Thus each of the terms in the expansion of $(a + b + c)^7$ is divisible by abc.

4. Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that

$$a_1 < a_2 < a_3 < a_4 < a_5,$$

$$b_1 > b_2 > b_3 > b_4 > b_5.$$

   (i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \leq j \leq 5$, is at least 6.

   (ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

Solution:

(i) Fix any pair $\{a_j, b_j\}$. We have $a_1 < a_2 < \cdots < a_{j-1} < a_j$ and $b_j > b_{j+1} > \cdots > b_5$. Thus there are $j-1$ numbers smaller than a_j and $5-j$ numbers smaller than b_j. Together they account for $j-1+5-j=4$ distinct numbers smaller than a_j as well as b_j. Hence the larger of a_j and b_j is at least 6.

(ii) The first part shows that the larger numbers in the pairs $\{a_j, b_j\}$, $1 \le j \le 5$, are $6, 7, 8, 9, 10$ and the smaller numbers are$1, 2, 3, 4, 5$. This implies that

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5|$$
$$= 10 + 9 + 8 + 7 + 6 - (1 + 2 + 3 + 4 + 5) = 25.$$

5. The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

Solution: Let the lengths of the arcs XY, UV, EF, GH be respectively p, q, r, s. We also the following notations: (See figure)

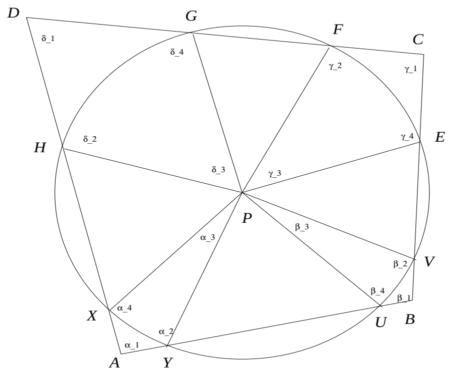

Fig. 3

$\angle XAY = \alpha_1$, $\angle AYP = \alpha_2$, $\angle YPX = \alpha_3$, $\angle PXA = \alpha_4$, $\angle UBY = \beta_1$, $\angle BVP = \beta_2$, $\angle VPU = \beta_3$, $\angle PUB = \beta_4$, $\angle ECF = \gamma_1$, $\angle CFP = \gamma_2$, $\angle FPE = \gamma_3$, $\angle PEC = \gamma_4$, $\angle GDH = \delta_1$, $\angle DHP = \delta_2$, $\angle HPG = \delta_3$, $\angle PGD = \delta_4$.

We observe that

$$\sum \alpha_j = \sum \beta_j = \sum \gamma_j = \sum \delta_j = 2\pi.$$

It follows that

$$\sum (\alpha_j + \gamma_j) = \sum (\beta_j + \delta_j).$$

On the other hand, we also have $\alpha_2 = \beta_4$ since $PY = PU$. Similarly we have other relations: $\beta_2 = \gamma_4$, $\gamma_2 = \delta_4$ and $\delta_2 = \alpha_4$. It follows that

$$\alpha_1 + \alpha_3 + \gamma_1 + \gamma_3 = \beta_1 + \beta_3 + \delta_1 + \delta_3.$$

But $p + r = q + s$ implies that $\alpha_3 + \gamma_3 = \beta_3 + \delta_3$. We thus obtain

$$\alpha_1 + \gamma_1 = \beta_1 + \delta_1.$$

Since $\alpha_1 + \gamma_1 + \beta_1 + \delta_1 = 360°$, it follows that $ABCD$ is a cyclic quadrilateral.

6. For any natural number $n > 1$, prove the inequality:

$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}.$$

Solution: We have $n^2 < n^2 + 1 < n^2 + 2 < n^2 + 3 \cdots < n^2 + n$. Hence we see that

$$\begin{aligned} \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} &> \frac{1}{n^2+n} + \frac{2}{n^2+n} + \cdots + \frac{n}{n^2+n} \\ &= \frac{1}{n^2+n}(1+2+3+\cdots n) = \frac{1}{2}. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} &< \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \\ &= \frac{1}{n^2}(1+2+3+\cdots n) = \frac{1}{2} + \frac{1}{2n}. \end{aligned}$$

7. Find all integers a, b, c, d satisfying the following relations:

(i) $1 \le a \le b \le c \le d$;

(ii) $ab + cd = a + b + c + d + 3$.

Solution: We may write (ii) in the form

$$ab - a - b + 1 + cd - c - d + 1 = 5.$$

Thus we obtain the equation $(a-1)(b-1) + (c-1)(d-1) = 5$. If $a - 1 \ge 2$, then (i) shows that $b - 1 \ge 2$, $c - 1 \ge 2$ and $d - 1 \ge 2$ so that $(a-1)(b-1) + (c-1)(d-1) \ge 8$. It follows that $a - 1 = 0$ or 1.

If $a - 1 = 0$, then the contribution from $(a-1)(b-1)$ to the sum is zero for any choice of b. But then $(c-1)(d-1) = 5$ implies that $c - 1 = 1$ and $d - 1 = 5$ by (i). Again (i) shows that $b - 1 = 0$ or 1 since $b \le c$. Taking $b - 1 = 0$, $c - 1 = 1$ and $d - 1 = 5$ we get the solution $(a, b, c, d) = (1, 1, 2, 6)$. Similarly, $b-1 = 1$, $c-1 = 1$ and $d-1 = 5$ gives $(a, b, c, d) = (1, 2, 2, 6)$.

In the other case $a - 1 = 1$, we see that $b - 1 = 2$ is not possible for then $c - 1 \ge 2$ and $d - 1 \ge 2$. Thus $b - 1 = 1$ and this gives $(c-1)(d-1) = 4$. It follows that $c - 1 = 1$, $d - 1 = 4$ or $c - 1 = 2$, $d - 1 = 2$. Considering each of these, we get two more solutions: $(a, b, c, d) = (2, 2, 2, 5), (2, 2, 3, 3)$.

It is easy to verify all these four quadruples are indeed solutions to our problem.
