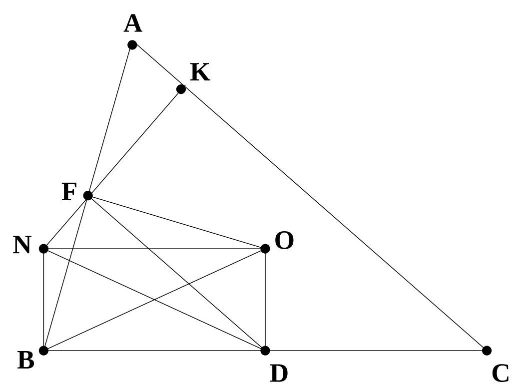
Solutions to CRMO-2008 Problems

1. Let ABC be an acute-angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N. Prove that ND is equal to the circum-radius of ABC. [15]

Solution: Let O be the circum-centre of ABC. Join OD, ON and OF. We show that $BDON$ is a rectangle. It follows that $DN = BO = R$, the circum-radius of ABC.



Observe that $\angle NBC = \angle NKC = 90°$. Hence $BCKN$ is a cyclic quadrilateral. Thus $\angle KNB = 180° - \angle BCA$. But $\angle BOA = 2\angle BCA$ and OF bisects $\angle BOA$. Hence $\angle BOF = \angle BCA$. We thus obtain

$$\angle FNB + \angle BOF = \angle KNB + \angle BCK = 180°.$$

This implies that B, O, F, N are con-cyclic. Hence $\angle BFO = \angle BNO$. But observe that $\angle BFO = 90°$ since OF is perpendicular to AB. Thus $\angle BNO = 90°$. Since NB and OD are perpendicular to BC, it follows that $BDON$ is a rectangle.

Alternate Solution: We can also get the conclusion using trigonometry. Observe that $\angle NFB = \angle AFK = 90° - \angle A$; and $\angle BNF = 180° - \angle B$ since $BCKN$ is a cyclic quadrilateral. Using the sine-rule in the triangle BFN,

$$\frac{NB}{\sin \angle NFB} = \frac{BF}{\sin \angle BFN}.$$

This reduces to

$$NB = \frac{c}{2}\frac{\cos A}{\sin C} = R\cos A.$$

But $BD = a/2 = R\sin A$. Thus

$$ND^2 = NB^2 + BD^2 = R^2.$$

This gives $ND = R$.

2. Prove that there exist two infinite sequences $\langle a_n\rangle_{n\geq 1}$ and $\langle b_n\rangle_{n\geq 1}$ of positive integers such that the following conditions hold simultaneously:

   (i) $1 < a_1 < a_2 < a_3 < \cdots$;

   (ii) $a_n < b_n < a_n^2$, for all $n \geq 1$;

   (iii) $a_n - 1$ divides $b_n - 1$, for all $n \geq 1$;

   (iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \geq 1$.

[19]

Solution: Let us look at the problem of finding two positive integers a, b such that $1 < a < b < a^2$, $a-1$ divides $b-1$ and a^2-1 divides b^2-1. Thus we have

$$b-1 = k(a-1), \quad \text{and} \quad b^2-1 = l(a^2-1).$$

Eliminating b from these equations, we get

$$(k^2-l)a = k^2-2k+l.$$

Thus it follows that

$$a = \frac{k^2-2k+l}{k^2-l} = 1 - \frac{2(k-l)}{k^2-l}.$$

We need a to be an integer. Choose $k^2-l=2$ so that $a = 1+l-k = k^2-k-1$ and $b = k(a-1)+1 = k^3-k^2-2k+1$. We want $a>1$ which is assured if we choose $k \geq 3$. Now $a<b$ is equivalent to $(k^2-1)(k-2)>0$ which again is assured once $k \geq 3$. It is easy to see that $b<a^2$ is equivalent to $k(k^3-3k^2+4)>0$ and this is also true for all $k \geq 3$. Thus we define

$$\begin{aligned} a_n &= (n+2)^2-(n+2)-1 = n^2+3n+1, \\ b_n &= (n+2)^3-(n+2)^2-2(n+2)+1 = n^3+5n^2+6n+1, \end{aligned}$$

for $n \geq 1$. Then we see that

$$1 < a_n < b_n < b_n^2,$$

for all $n \geq 1$. Moreover

$$a_n - 1 = n(n+3), \quad b_n - 1 = n(n+3)(n+2)$$

and

$$a_n^2 - 1 = n(n+3)(n+1)(n+2), \quad b_n^2 - 1 = n(n+3)(n+2)(n+1)(n^2+4n+2).$$

Thus we have a pair of desired sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

3. Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that:

$$\text{(i)} \ \ 0 < 3ab \leq 1 \quad \text{and} \quad \text{(ii)} \ \ b \geq \sqrt{3}.$$

[19]

Solution: Let α, β, γ be the roots of the given equation. We have

$$\alpha+\beta+\gamma = \frac{1}{a}, \quad \alpha\beta+\beta\gamma+\gamma\alpha = \frac{b}{a}, \quad \alpha\beta\gamma = \frac{1}{a}.$$

It follows that a, b are positive. We thus obtain

$$\frac{3b}{a} = 3(\alpha\beta+\beta\gamma+\gamma\alpha) \leq (\alpha+\beta+\gamma)^2 = \frac{1}{a^2},$$

which gives $0 < 3ab \leq 1$. Moreover

$$\begin{aligned} \frac{b^2}{a^2} &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \frac{2}{a^2}. \end{aligned}$$

Thus

$$\frac{b^2 - 2}{a^2} = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 \geq \frac{1}{3}(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \frac{b^2}{3a^2}.$$

This implies that $3(b^2 - 2) \geq b^2$ or $b^2 \geq 3$. Hence $b \geq \sqrt{3}$, the conclusion follows.

4. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0,1,2,3 occurs at least once in them. [14]

Solution: We observe that $0 + 1 + 2 + 3 = 6$. Hence the remaining two digits must account for for the sum 4. This is possible with $4 = 0 + 4 = 1 + 3 = 2 + 2$. Thus we see that the digits in any such 6-digit number must be from one of the collections: $\{0, 1, 2, 3, 0, 4\}$, $\{0, 1, 2, 3, 1, 3\}$ or $\{0, 1, 2, 3, 2, 2\}$.

Consider the case in which the digits are from the collection $\{0, 1, 2, 3, 0, 4\}$. Here 0 occurs twice and the digits 1,2,3,4 occur once each. But 0 cannot be the first digit. Hence the first digit must be one of 1,2,3,4. Suppose we fix 1 as the first digit. Then the number of 6-digit numbers in which the remaining 5 digits are 0,0,2,3,4 is $5!/2! = 60$. Same is the case with other digits: 2,3,4. Thus the number of 6-digit numbers in which the digits 0,1,2,3,0,4 occur is $60 \times 4 = 240$.

Suppose the digits are from the collection $\{0, 1, 2, 3, 1, 3\}$. The number of 6-digit numbers beginning with 1 is $5!/2! = 60$. The number of those beginning with 2 is $5!/(2!)(2!) = 30$ and the number of those beginning with 3 is $5!/2! = 60$. Thus the total number in this case is $60 + 30 + 60 = 150$. Alternately, we can also count it as follows: the number of 6-digit numbers one can obtain from the collection $\{0, 1, 2, 3, 1, 3\}$ with 0 also as a possible first digit is $6!/(2!)(2!) = 180$; the number of 6-digit numbers one can obtain from the collection $\{0, 1, 2, 3, 1, 3\}$ in which 0 is the first digit is $5!/(2!)(2!) = 30$. Thus the number of 6-digit numbers formed by the collection $\{0, 1, 2, 3, 1, 3\}$ such that no number has its first digit 0 is $180 - 30 = 150$.

Finally look at the collection $\{0, 1, 2, 3, 2, 2\}$. Here the number of of 6-digit numbers in which 1 is the first digit is $5!/3! = 20$; the number of those having 2 as the first digit is $5!/2! = 60$; and the number of those having 3 as the first digit is $5!/3! = 20$. Thus the number of admissible 6-digit numbers here is $20 + 60 + 20 = 100$. This may also be obtained using the other method of counting: $6!/3! - 5!/3! = 120 - 20 = 100$.

Finally the total number of 6-digit numbers in which each of the digits 0,1,2,3 appears at least once is $240 + 150 + 100 = 490$.

5. Three real numbers $a, b,$ c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c. [17]

Solution: Since $20, b, c$ are in harmonic progression, we have

$$\frac{1}{20} + \frac{1}{c} = \frac{2}{b},$$

which reduces to $bc + 20b - 40c = 0$. This may also be written in the form

$$(40 - b)(c + 20) = 800.$$

Thus we must have $20 < b < 40$ or, equivalently, $0 < 40 - b < 20$. Let us consider the factorisation of 800 in which one term is less than 20:

$$(40 - b)(c + 20) = 800 = 1 \times 800 = 2 \times 400 = 4 \times 200$$
$$= 5 \times 160 = 8 \times 100 = 10 \times 80 = 16 \times 50.$$

We thus get the pairs

$$(b, c) = (39, 780), \quad (38, 380), (36, 180), (35, 140), (32, 80), (30, 60), (24, 30).$$

Among these 7 pairs, we see that only 5 pairs $(39, 780)$, $(38, 380)$, $(36, 180)$, $(35, 140)$, $(30, 60)$ fulfill the condition of divisibility: b divides c. Thus there are 5 triples satisfying the requirement of the problem.

6. Find the number of all integer-sided *isosceles obtuse-angled triangles* with perimeter 2008. [16]

Solution: Let y, x, x be the sides of the triangle, where these are positive integers. Since we are looking for obtuse-angled triangles, $y > x$. Moreover, $y + 2x = 2008$ shows that y is even. But $y < 1004$ by triangle inequality. Thus triples are $(y, x, x) = (1002, 503, 503)$, $(1000, 504, 504)$, $(998, 505, 505)$, and so on. The general form is $(y, x, x) = (1004 - 2k, 502 + k, 502 + k)$, where $k = 1, 2, 3, \ldots, 501$. But the condition that the triangle is obtuse leads to

$$(1004 - 2k)^2 > 2(502 + k)^2.$$

This simplifies to

$$502^2 + k^2 - 6(502)k > 0.$$

Solving this quadratic inequality for k, we see that

$$k < 502(3 - 2\sqrt{2}), \quad \text{or} \quad k > 502(3 + 2\sqrt{2}).$$

Since $k \leq 501$, we can rule out the second possibility. Thus $k < 502(3 - 2\sqrt{2})$, which is approximately 86.1432. We conclude that $k \leq 86$. Thus we get 86 triangles

$$(y, x, x) = (1004 - 2k, 502 + k, 502 + k), \quad k = 1, 2, 3, \ldots, 86.$$

The last obtuse triangle in this list is: (832,588,588). (It is easy to check that $832^2 - 588^2 - 588^2 = 736 > 0$, where as $830^2 - 589^2 - 589^2 = -4942 < 0$.)

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