Regional Mathematical Olympiad-2016

Time: 3 hours **October 23, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be an isosceles triangle with $AB = AC$. Let Γ be its circumcircle and let O be the cetre of Γ. Let CO meet Γ in D. Draw a line parallel to AC through D. Let it intersect AB at E. Suppose $AE : EB = 2 : 1$. Prove that ABC is an equilateral triangle.

2. Let a, b, c be positive real numbers such that
$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$
Prove that $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$.

3. The present ages in years of two brothers A and B, and their father C are three distinct positive integers a, b, and c respectively. Suppose $\frac{b-1}{a-1}$ and $\frac{b+1}{a+1}$ are two consecutive integers, and $\frac{c-1}{b-1}$ and $\frac{c+1}{b+1}$ are two consecutive integers. If $a+b+c \leq 150$ determine a, b and c.

4. A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

5. Let ABC be a triangle, AD an altitude and AE a median. Assume B, D, E, C lie in that order on the line BC. Suppose the incentre of triangle ABE lies on AD and the incentre of ADC lies on AE. Find, with proof, the angles of triangle ABC.

6. (i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.

   (ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

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